zero and one is a best reply.

**Exercise 85.** Find the player 2’s best-reply function!

**Answer:** Use the same procedure as for player 1, defining \( q^* = \frac{(d-b)}{(a-c)+(d-b)} \).

Player 2’s best-reply function is also displayed in Figure 22.2.

### 22.6.2.3. Equilibrium

The Nash equilibrium is where both players are on their best replies, i.e. the intersection of the red and the green curves in Figure 22.2. Notice that this equilibrium requires both players to randomize between their two choices. Also note that the equilibrium probabilities depend on the payoffs, A, B, C, D as well as a, b, c and d.

### 22.6.2.4. Existence of equilibrium

A fundamental result in game theory, due to John Nash, is that any game with (i) a finite number of players and (ii) where each player has a finite number of strategies has at least one Nash equilibrium. Sometimes, as in the football penalty game, there doesn’t exist an equilibrium in pure strategies. But then there is always at least one equilibrium in mixed strategies.

### 22.6.3. A rationale for mixed strategy equilibrium (Difficult)

Many people are skeptical about mixed strategy equilibria. It appears unlikely that mixed equilibria are good predictions of how people behave in the real world. Thus, a substantial part of economic analysis is put into question. One problem with mixed strategy equilibria is that they seem to be bad descriptions of real-world decision making processes. “Players” including governments, firms
and regular people acting on their own behalf don’t usually throw dice when making important decisions. And why should they? Why would such players follow the particular probability distribution, prescribed by game theory, when there are several pure strategies that give the same expected payoffs? People may rather be inclined to just choose one or the other pure strategy, without much thought. And there is no obvious reason why an individual’s tendency to select one equally good strategy over the other would coincide with the probabilities prescribed by game theory. The answer that game theory would have us believe is that players must follow their mixed strategies, not because it is necessary for maximizing their own utilities, but because it is necessary for making the other players indifferent between the pure strategies they are supposed to use with positive probability. That is clearly not a very convincing argument.

Fortunately, there are better arguments. But then we need to go a bit outside of the narrow game we are studying. Remember that the formal games we analyze are intended to be simplified descriptions of the world, removing some of the complexity that makes real-world situations so difficult to understand. For instance, the payoffs we attach to different outcomes are only meant to describe the “typical” payoff associated with that outcome. In the real world people differ; some people get a slightly higher and some get a slightly lower payoff than the “typical” payoff. Thus, put in a situation where all people would consider two alternatives almost equally good, some of them will actually slightly prefer one alternative and others will slightly prefer the other.

The big question is then if there is any reason to believe that mixed strategies can be interpreted as the frequency distribution of choices in the population? The so-called purification theorem demonstrates that the answer is “yes.” To illustrate the purification theorem, we will consider the so-called matching pennies game (a.k.a the football penalty game).
Matching Pennies  The matching pennies game is described by the payoff matrix in Table 22.15. This game has two players, called 1 and 2. Both players have two strategies, called H (head) and T (tails). Player 1 wins if the players choose the same strategies, i.e. H and H or T and T, and player 2 wins otherwise. Both players get payoff 1 when winning and -1 when losing.

Table 22.15.: Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1, -1</td>
<td>-1,1</td>
</tr>
<tr>
<td>T</td>
<td>-1,1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

This game can for example be viewed as a stylized representation of a penalty kick in football. The goalie (player 1) wins if he defends the same side that the shooter kicks.

In this game there is no pure strategy equilibrium. There is however a (unique) mixed strategy equilibrium, requiring both players to select their two strategies with equal probabilities. But, again, why would they?

Matching Pennies with incomplete information  Remember that Table 22.15 is a simplified description of the situation. Different people acting in the role of player 1 have slightly different payoffs. It may for example be that they get some extra utility or disutility from selecting H. Let us say they get \( \varepsilon \cdot x_1 \). The first factor, \( \varepsilon \), which is a small positive number, is the same for all people who act in role 1. It is also known by all people, including those who act in role 2. The second factor, \( x_1 \), differ between different individuals. And, while every person acting in the role of player 1 knows his own type (his own value \( x_1 \)) the opponent does not. We say that there is incomplete information. Let us for simplify assume that \( x_1 \) is uniformly distributed on \([-1, 1]\).\(^3\) Similarly, every person acting

\(^3\)This means that the probability that \( x_1 \geq y \) is given by \( \frac{1-y}{1+y} = \frac{1}{2} - \frac{1}{2} \cdot y. \)
in the role of player 2 gets the extra payoff $\varepsilon \cdot x_2$ when selecting $H$. These payoffs are illustrated in Table 22.16.

Table 22.16.: Matching Pennies with incomplete information

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$1 + \varepsilon \cdot x_1$, $-1 + \varepsilon \cdot x_2$</td>
<td>$-1 + \varepsilon \cdot x_1, 1$</td>
</tr>
<tr>
<td>$T$</td>
<td>$-1, 1 + \varepsilon \cdot x_2$</td>
<td>$1, -1$</td>
</tr>
</tbody>
</table>

Assume now that every person acting on one of the two roles will choose one of the two actions with certainty. Nobody uses a mixed strategy. Still, however, different people acting in the role of player 1 may prefer different actions, depending on their types (i.e. depending on their $x_1$). Similarly, different people acting in the role of player 2 will make different decisions, depending on their types. To describe these different choices we let $p_1$ denote the share of people acting as player 1 who select $H$. Similarly, $p_2$ denotes the share of people acting as player 2 who select $H$.

Since a person in the role of player 1 does not know what type his opponent is, the opponent’s choice will appear as a random variable, with the probability of $H$ being $p_2$. Then, the expected utility of choosing $H$, for a person in the role of player 1, characterized by $x_1$, is given by

$$EU_1(H) = (1 + \varepsilon \cdot x_1) \cdot p_2 + (-1 + \varepsilon \cdot x_1) \cdot (1 - p_2).$$

Similarly, the expected utility of choosing $T$, is given by

$$EU_1(T) = (-1) \cdot p_2 + (1) \cdot (1 - p_2).$$

Thus, a person acting in the role of player 1 will select $H$ if and only if

$$x_1 \geq x_1 = \frac{1 - 2 \cdot p_2}{\varepsilon}.$$

Since $x_1$ is uniformly distributed over $[1, 1]$, the probability that a person in the role of player 1 selects $H$ is given by $p_1 = \frac{1 - x_1}{1 - (1 - 1)}$. 460
Thus, substituting for $a_1$, we can compute the population share of players 1 selecting $H$ as a function of the population share of players 2 selecting $H$ as:

$$p_1 = \frac{1}{2} - \frac{1 - 2 \cdot p_2}{\varepsilon}.$$

**Exercise 86.** Show that the probability that a person in the role of player 2 selects $H$ is given by

$$p_2 = \frac{1}{2} + \frac{1 - 2 \cdot p_1}{\varepsilon}.$$

We are thus left with a system of two linear equations in two unknowns and it is easy to see that the solution is $p_1 = p_2 = \frac{1}{2}$. The point is, of course, that this is the same distribution as prescribed by the mixed strategy equilibrium of the original game.

**Interpretations** Notice that people acting in the role of player 1 need to be informed about $p_2$ to determine their own best choice. There are two alternative assumptions to motivate why they may possess this knowledge. First, they may observe that half of players 2 have selected $H$ historically and therefore assume that their current opponent will select $H$ with these probabilities. An alternative interpretation is that player $i$ is informed about the distribution of the unobservable $x_j$ in the population and that player $i$ therefore can compute the equilibrium value of $p_j$.

An alternative version of the story told here is that we are studying the same two people playing the game over and over, but that their utilities vary slightly over time. The formal analysis would be the same.

**Purification theorem** We started out with a normal form game with a Nash equilibrium in mixed strategies. We constructed a new “perturbed” game of incomplete information that is very similar to the original game. The perturbed game has the same set of players.
22. Static Games

and the same set of strategies. But, the players’ payoffs are stochastic and vary slightly around their means. We assumed that the players only know their own payoffs. We demonstrated that there is a Bayesian-Nash pure-strategy equilibrium of the incomplete information game that implies the same probability distribution over outcomes as the mixed strategy equilibrium of the original game.

The purification theorem by Harsanyi\(^4\) states that the same procedure can be used for any mixed strategy Nash equilibrium for almost all (finite) normal form games.\(^5\) It follows that we may interpret the mixed strategy equilibrium of a game as an approximation of the population distribution of strategies in a slightly more complicated game with incomplete information.

22.7. Basic idea of an equilibrium

An equilibrium can be described as a common understanding of how people will behave in a certain situation. Sometimes there is such an understanding. An example is that we all know that all drivers in Sweden keep to the right. It is important to note that in order for such a common understanding to arise, every person must have an incentive to indeed do what is expected from him. Otherwise it would be foolish to expect him to do it.

It could not be an equilibrium that 90% of all drivers keep to the right and 10% keep to the left, since then the people who are presumed to drive on the left side would be better off switching lane. They would be involved in fewer accidents.

That there is a common understanding of how people behave


\(^5\)The fact that the probability distribution over outcomes is exactly the same is not typical. In most games, the equilibrium distribution of the incomplete game will depend on \(\varepsilon\) and be slightly different from the mixed strategy equilibrium of the original game. But, as \(\varepsilon \to 0\) the two distributions will converge.