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# Auctions

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# Auctions

- Examples
  - Antiques, fine arts
  - Houses, apartments, land
  - Government bonds, bankrupt assets
  - Government contracts (roads)
  - Radio frequencies

# Auctions

## Why use auction?

- Seller's goal
  - Maximize revenues (you selling your apartment)
  - Efficient use (Government selling radio spectrum)
- Problem
  - Seller doesn't know what people are willing to pay
    - What is the highest valuation?
    - Who has it?
- Solution
  - Buyer *claiming* highest valuation gets the good
  - And will pay accordingly
- Auction = Mechanism to extract information

# Auctions

## But are auctions a good solution?

- Efficiency
  - IF: people really "tell the truth" = bid their valuations
  - THEN: good will be allocated correctly
- Revenues
  - IF: people really "tell the truth" = bid their valuations
  - THEN: price will be high (efficiency & extract WTP)
- Question: Do people "tell the truth"?
  - Need to study bidding behavior

# Auctions

- Bidding behavior turns out to depend on:
  - Exact rules of the auction (Auction design)
  - How buyer's valuations are related (Type of uncertainty)

# Auctions

## 4 Designs

- Sealed bid, second price ("Vickrey")
  - Simultaneous
  - Winner pays second bid
- Sealed bid, first price
  - Simultaneous
  - Winner pays own bid
- English ("open cry")
  - Sequential + perf. info
  - Ascending bids
- Dutch
  - Sequential + perf. info
  - Descending offers

**We will only study most common**

# Auctions

## Types of Uncertainty

- Private value
  - Different buyers have different values
- Common value
  - Same value to all buyers
  - But different buyers have different information

**We will only study private value**

# English Auction

# English Auction

- Assume
  - One indivisible unit of the good
  - Two bidders
- Information
  - Bidders get to know own valuations, $v_1$ and $v_2$
  - Then the bidding game starts
- Bidding rules: a simple model
  - Players take turns bidding
  - Whenever one player does not bid at least €1 more, the good is sold to the current bid

# English Auction

- Outcome
  - Winner = Highest bidder
  - Price = Highest bid

# Second-Price Sealed-Bids

- Utility

$$u_i = \begin{cases} v_i - b_j & \text{if winning} \\ 0 & \text{otherwise} \end{cases}$$

# English Auction

- Define: "marginal increases strategy" for i
  - If current bid < valuation, raise by €1
  - If current bid > valuation, stop bidding
- Formally
  - IF: $b_{jt-1} + 1 \leq v_i$, THEN: bid $b_{it} = b_{jt-1} + 1$
  - IF: $b_{jt-1} + 1 > v_i$, THEN: stop bidding
- Claim
  - This strategy is optimal (actually, dominant)

# English Auction

- Sketch of proof
  - **If $b_{2t-1} < v_1$**
    - **Outbid**: Positive utility with (weakly) positive probability
    - **Withdraw**: $u_1 = 0$ for sure
  - **If $b_{2t-1} \geq v_1$**
    - **Withdraw**: $u_1 = 0$ for sure
    - **Outbid**: Negative utility with (weakly) positive probability
  - Note - dominance
    - Above strategy optimal
    - no matter how $b_{2t-1}$ selected

# English Auction

- Outcome
  - "Truth telling" (people increase the price as long as price is below their valuation)
  - Q: Who gets the good?

# English Auction

- Outcome
  - "Truth telling"
  - Efficiency
    - Bidder with highest valuation wins the good
  - Q: Who gets the surplus?

# English Auction

- Outcome
  - "Truth telling"
  - Efficiency
    - Bidder with highest valuation wins the good
  - Surplus-sharing
    - p = SHV   (sometimes p = SHV + 1)

Not this year

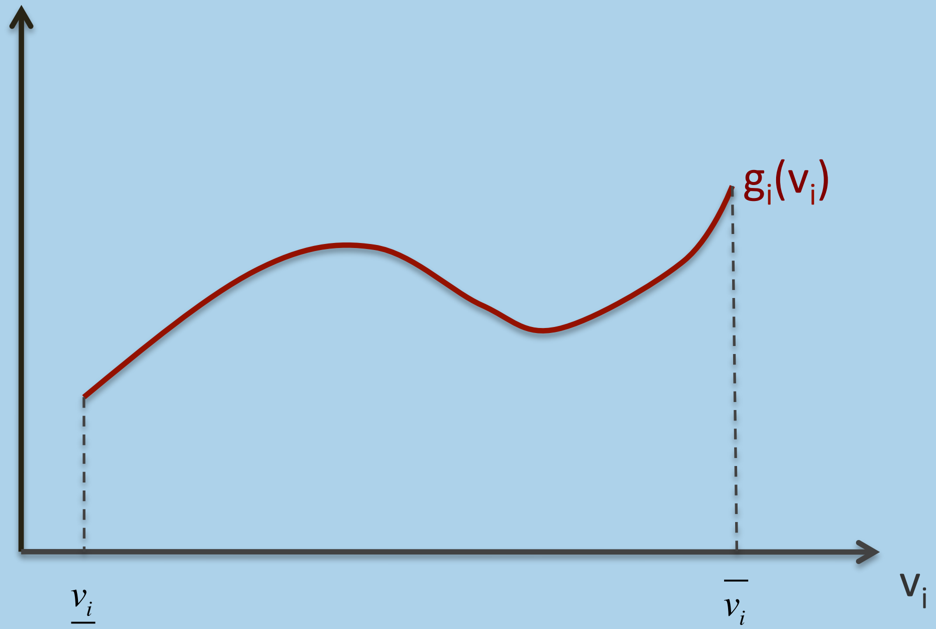
# Second-Price Sealed-Bids Auction

# Second-Price Sealed-Bids

- Assume
  - One indivisible unit of the good
  - Two bidders

# Second-Price Sealed-Bids

- Modeling asymmetric information
  - Player i's valuations lies in interval $[\underline{v}_i, \overline{v}_i]$
  - Probability density $g_i$



# Second-Price Sealed-Bids

- Timing
  - Bidders get to know own valuations, $v_1$ and $v_2$
  - Simultaneous bids, $b_1$ and $b_2$
- Strategy
  - Function: $B_i(v_i)$

# Second-Price Sealed-Bids

- Outcome
  - Winner = Highest bidder
  - Price = Second highest bid

# Second-Price Sealed-Bids

- Utility

$$u_i = \begin{cases} v_i - b_j & b_i < b_j \\ \frac{1}{2}\left(v_i - b_j\right) \quad if & b_i = b_j \\ 0 & b_i > b_j \end{cases}$$

  – Note: risk-neutrality

# Second-Price Sealed-Bids

- Claim
  - $B^*_i(v_i) = v_i$ $i=1,2$ is a BNE
  - "truth-telling"
  - Actually: Weakly dominating strategy

# Second-Price Sealed-Bids

- If (for some reason) $b_1 < v_2$
  - Q: Optimal $b_2$ ?

# Second-Price Sealed-Bids

- If (for some reason) $b_1 < v_2$
  - Any $b_2 > b_1$ maximizes 2's utility
    - Gets the good
    - Pays $b_1$
    - $U = v_2 - b_1$ $\quad$ ($b_2 < b_1 \Rightarrow U = 0$)
  - Example: $b_2 = v_2$

# Second-Price Sealed-Bids

- If (for some reason) $b_1 > v_2$
  - Q: Optimal $b_2$ ?

# Second-Price Sealed-Bids

- If (for some reason) $b_1 > v_2$
  - Any $b_2 < b_1$ maximizes 2's utility
    - Does not get the good
      (has to pay more than value to get it)
  - Example: $b_2 = v_2$

# Second-Price Sealed-Bids

- If (for some reason) $b_1 = v_2$
  - Q: Optimal $b_2$ ?

# Second-Price Sealed-Bids

- If (for some reason) $b_1 = v_2$
  - Any $b_2$ maximizes 2's utility
    - $b_2 > b_1$ => get the good, pay $b_1 = v_2$
    - $b_2 < b_1$ => don't get good
    - $b_2 = b_1$ => random
  - Example: $b_2 = v_2$

# Second-Price Sealed-Bids

- Conclusion
  - Independent of $b_1$
  - Bidding $b_2 = v_2$ maximizes utility
- Note
  - Weakly dominant strategy
  - Also: Distribution over $v_1$ does not matter

# Second-Price Sealed-Bids

- Outcome
  - "Truth telling"
  - Q: Who gets the good?

# Second-Price Sealed-Bids

- Outcome
  - "Truth telling"
  - Efficiency
    - Bidder with highest valuation wins the good
  - Q: Who gets the surplus?

# Second-Price Sealed-Bids

- Outcome
  - "Truth telling"
  - Efficiency
    - Bidder with highest valuation wins the good
  - Surplus-sharing
    - p = SHV

# Auctions

- Intuition for “truth telling”
  – Own bid does not affect the price you pay

# First-Price Sealed-Bids Auction

# First-Price Sealed-Bids

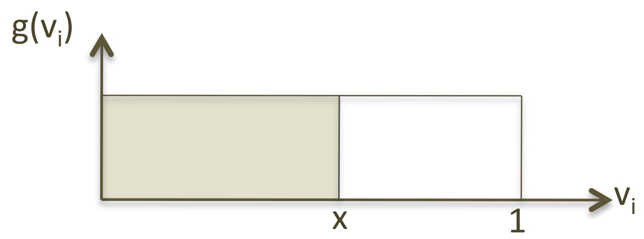
- Rules
  - Simultaneous bids (= sealed bids)
  - Winner pays his bid  (= first price)

# First-Price Sealed-Bids

- Trade-off
  - Higher bid → Higher probability of winning
  - Higher bid → Higher price

# First-Price Sealed-Bids

- Simplification
  - Two bidders: $v_1, v_2$
  - $v_i$ uniformly distributed over [0, 1]




prob($v_i$ < x) = x

# First-Price Sealed-Bids

- Payoff = expected utility
  - $E\pi_1(b_1) = (v_1 - b_1)\ Pr(win) + 0\ Pr(loose)$
  - $E\pi_1(b_1) = (v_1 - b_1)\ Pr(b_1 > b_2)$

**Depends on**

$b_1$ = own choice

$b_2$ = random variable

# First-Price Sealed-Bids

- Payoff = expected utility
  - $E\pi_1(b_1) = (v_1 - b_1)\ Pr(win) + 0\ Pr(loose)$
  - $E\pi_1(b_1) = (v_1 - b_1)\ Pr(b_1 > b_2)$

We need to compute probability that $b_2 < b_1$

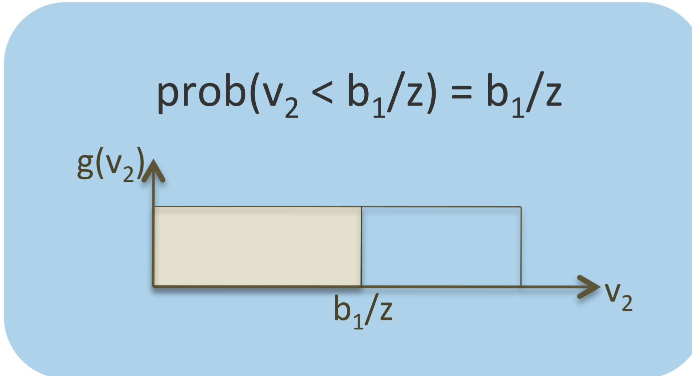
# First-Price Sealed-Bids

- Payoff = expected utility
  - $E\pi_1(b_1) = (v_1 - b_1)\ Pr(win) + 0\ Pr(loose)$
  - $E\pi_1(b_1) = (v_1 - b_1)\ Pr(b_1 > b_2)$

Simplifying assumption:
$b_2 = z \cdot v_2$

# First-Price Sealed-Bids

- Payoff = expected utility
  - $E\pi_1(b_1) = (v_1 - b_1)\ \Pr(\text{win}) + 0\ \Pr(\text{loose})$
  - $E\pi_1(b_1) = (v_1 - b_1)\ \Pr(b_1 > b_2)$
  - $E\pi_1(b_1) = (v_1 - b_1)\ \Pr(b_1 > z \cdot v_2)$
  - $E\pi_1(b_1) = (v_1 - b_1)\ \Pr(v_2 < b_1/z)$
  - $E\pi_1(b_1) = (v_1 - b_1)\ (b_1/z)$



# First-Price Sealed-Bids

- What is 1's best reply?
  - $E\pi_1(b_1) = (v_1 - b_1)\,(b_1/z)$
  - FOC: $(-1)\,(b_1/z) + (v_1 - b_1)\,(1/z) = 0$

Utility if winning * Increased probability of winning

# First-Price Sealed-Bids

## Proof

- Assume
  - $B_2(v_2) = z\ v_2$
- What is 1's best reply?
  - $E\pi_1(b_1) = (v_1 - b_1)\ (b_1/z)$
  - FOC: $(-1)\ (b_1/z) + (v_1 - b_1)\ (1/z) = 0$

Decreased utility * probability of winning

# First-Price Sealed-Bids

## Proof

- Assume
  - $B_2(v_2) = z\ v_2$
- What is 1's best reply?
  - $E\pi_1(b_1) = (v_1 - b_1)\ (b_1/z)$
  - FOC: $- (b_1/z) + (v_1 - b_1)/z = 0$
  - Solve: $b_1 = ½ \cdot v_1$

# First-Price Sealed-Bids

## Proof

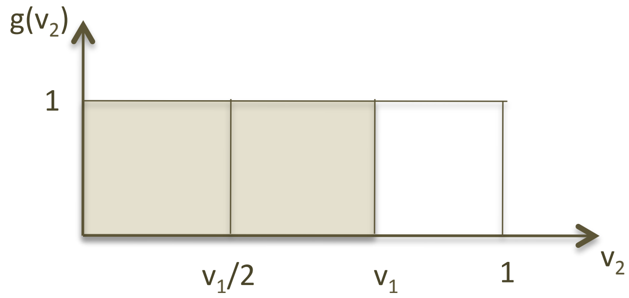
- Conclusion
  - IF: Bidder 2 uses a linear strategy: $B_2(v_2) = z \cdot v_2$
  - THEN: Best reply for bidder 1: $B_1(v_1) = ½ \cdot v_1$
- Note
  - Since $½ \cdot v_1$ is linear
  - Since players are symmetric
  - Both bidding $b_i = ½ \cdot v_i$ is a Nash equilibrium of a game where the strategy for each player is to choose some function $B_i(v_i)$.

# First-Price Sealed-Bids

- Interpretation
  - Why bid ½ v ?
- Answer 1
  - Optimal balance between
    - probability of winning
    - price in case of winning

# First-Price Sealed-Bids

- Interpretation
  - But why exactly ½ ?




- Answer 2
  - Assume you have highest valuation
  - Q: What is the expected second highest valuation?
  - Winner bids expected wtp of competitor
    => competitor no incentive to bid more

# First-Price Sealed-Bids

- Remark
  - With more bidders, expected second highest wtp is closer to highest wtp
  - Bid larger share of wtp
  - As $n \rightarrow \infty$ $b \rightarrow$ wtp

# First-Price Sealed-Bids

- Outcome
  - Q: Who gets the good?

# First-Price Sealed-Bids

- Outcome
  - Efficiency
    - Bidder with highest valuation wins the good
  - Q: Who gets the surplus?

# First-Price Sealed-Bids

- Outcome
  - Efficiency
    - Bidder with highest valuation wins the good
  - Surplus-sharing
    - p = ½ HV
  - Truth-telling?

# First-Price Sealed-Bids

- Outcome
  - Efficiency
    - Bidder with highest valuation wins the good
  - Surplus-sharing
    - p = ½ HV
  - "Sort of truth-telling"
    - Players actually reveal their valuation

# Game Theoretic "Details"

## Auction = Game of Incomplete Information

# Game of Incomplete Information

- Problem 1: Interdependent decisions
  - Adam's optimal bid depends on Ben's bid and vice versa
- Solution
  - Game theory

# Game of Incomplete Information

- Problem 2: Incomplete information
  - Buyers don't know each others' valuations
    - Adam not able to predict Ben's bid exactly
    - It depends on Ben's valuation of the object
  - How should Adam and Ben analyze the situation?

# Game of Incomplete Information

- Solution: Change definition of strategy
  - Strategy = Function prescribing bid for every possible valuation a player may have
- Example of strategy
  - IF wtp = $v_H$ THEN bid = $b_H$
  - IF wtp = $v_L$ THEN bid = $b_L$
- Then, players able to
  - Predict rival's *strategy*,
    even if uncertainty about type and *bid* remains
  - Maximize expected payoff

# Game of Incomplete Information

- But why are strategies functions?
  - Adam knows he has high valuation, $v_H$
  - Why should he choose strategy with instruction for $v_L$?
- Answer
  - Ben doesn't know Adam's valuation. Could be $v_H$ or $v_L$
  - Ben must consider
    - What would Adam bid if $v_H$
    - What would Adam bid if $v_L$
  - To predict Ben's bid, Adam must also consider what he himself would have bid in case of $v_L$

# Game of Incomplete Information

- Think of Adam's choice as two-step procedure

1. Find optimal bid for all possible valuations: $b^{Adam}(v_H)$ and $b^{Adam}(v_L)$
2. Select the relevant bid: $b^{Adam}(v_H)$

# Game of Incomplete Information

- Problem 2b: Incomplete information
  - Buyers don't know each others' valuations
- Solution: Change definition of payoff
  - Payoff = expected utility

# Game of Incomplete Information

## Bayesian Nash Equilibrium

- Pair of strategies $(b^{Adam}, b^{Ben})$ such that
- $b^{Adam}$ is a best reply to $b^{Ben}$
  - $b^{Adam}(v_H)$ maximizes Adam's expected utility
    - If Adam's valuation is $v_H$
    - Assuming Ben uses $b^{Ben}$
  - $b^{Adam}(v_L)$ maximizes Adam's expected utility
    - If Adam's valuation is $v_L$
    - Assuming Ben uses $b^{Ben}$
- $b^{Ben}$ is a best reply to $b^{Adam}$

# Most fundamental result of auction theory

# Fundamental result

**Note 1: No individual knows who has the highest valuation**

**Note 2: But if people play the auction game**
**⇒ person with highest valuation walks away with the good**

**No individual (even with the power of a dictator) could have implemented the efficient allocation, since nobody has sufficient information**

**But the <u>market mechanism</u> actually <u>solves</u> the <u>maximization problem</u>**

**May say the market <u>aggregates information</u>**
* **must use all the information to solve the max-problem**
* **despite the fact that it is scattered**

# Fundamental result

- Laboratory experiments
  - It works! (Vernon Smith)
  - Also double auctions
  - Even with “few” buyers and “few” sellers market quickly converges to competitive price
  - NB: must use laboratory to know people’s valuations

# Fundamental result

Sure, it is not perfect…

…there is also market failure…

- Coordination (mis-pricing; recessions)
- Double coincidence of wants (kidneys, apartments)
- Externalities (global warming; telecom)
- Public goods (R&D; legal system to enforce all contracts)
- Market power (medicines; district heating)
- Incomplete information (cars, insurance, labor, credit)

…and an uneven distribution of wealth

# Fundamental result

- But even public policies to correct market failure use markets to aggregate information
  - Cap and trade
  - Public procurement

# Comparison of Auction Designs (Revenues)

# Comparison of Designs

## Question 1

- Which auction gives the highest expected price?
  - FPSB (and Dutch): $p = \frac{1}{2}$ HV
  - English (and SPSB): $p = SHV$ Recall: $E(SHV) = \frac{1}{2}$ HV

# Comparison of Designs

## Answer 1

- Expected Revenue Equivalence Thm.
  (Vickrey, 1961)
  - All four auctions give the same *expected* price

# Comparison of Designs

## Question 2

- Is there any other way to sell the goods which would give a higher expected profit?
  - Lots of different possible ways
    - Bargaining
    - Other auction formats
    - Strange games

# Comparison of Designs

## Answer 2

- Generalization of Revenue Equivalence Thm
  - No!
  - This is example of “mechanism design” and uses the “revelation principle” (Leonid Hurwicz, Eric Maskin, Roger Myerson)